Quantum computation and Shor's factoring algorithm

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- We focus on second question: quantum algorithms

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- Efficient if $|C_n| \leq poly(n)$

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  2. Measurement gives $|x\rangle$ with probability $|\alpha_x|^2$

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- 2-qubit controlled-NOT: $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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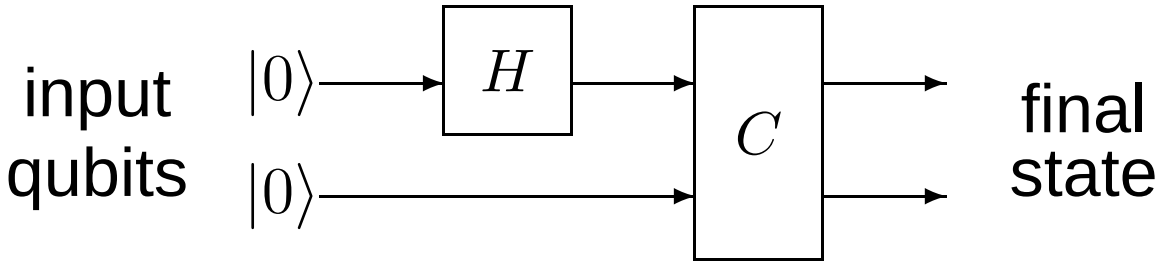

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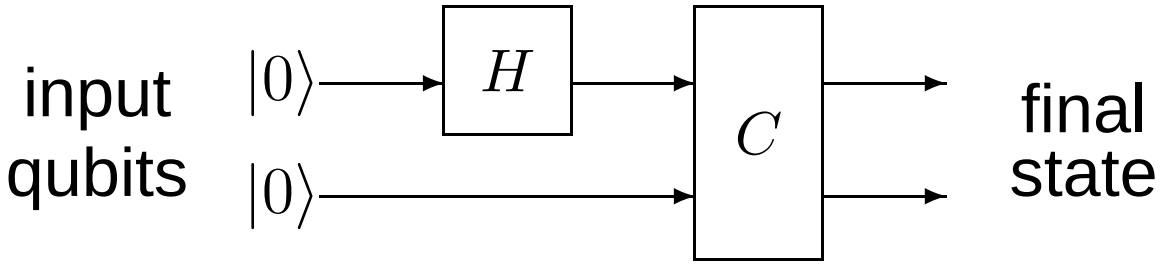

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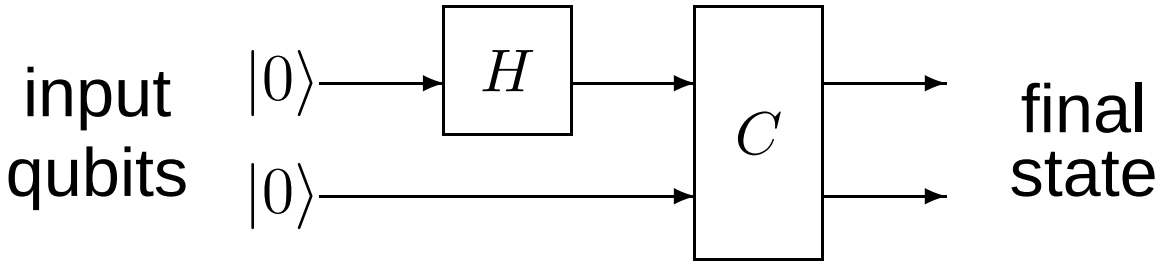

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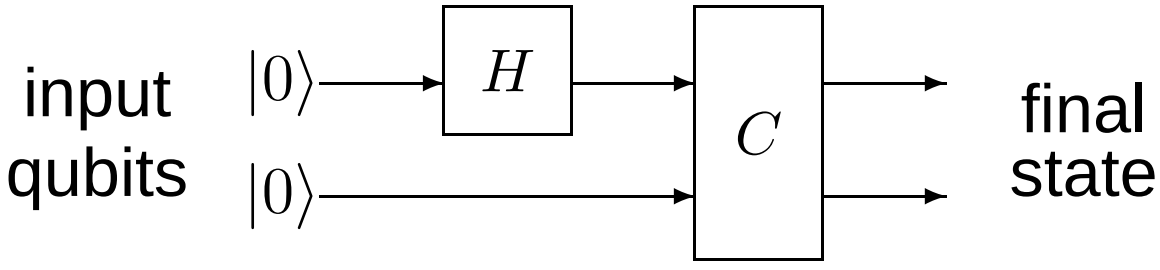

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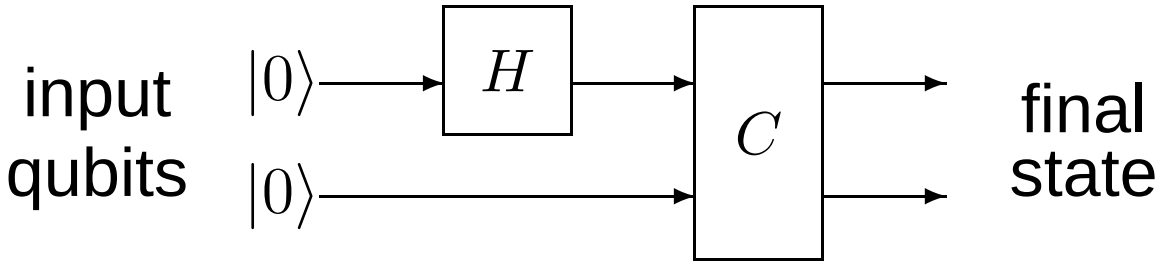

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- Efficient quantum computation: polynomial family

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- More tricks needed for successful quantum computation

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- This factor of N can be extracted using gcd-algorithm

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4. Apply quantum Fourier transform to 1st register
5. Measure 1st register and computer r

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- For our q, we choose the power of 2 in $(N^2, 2N^2]$

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- Since we can find r, we can factor!

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- Jones polynomial (Aharonov et al.)

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